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## Uniqueness and non-degeneracy of positive radial solutions of quasilinear Schrödinger equations

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### 1. Introduction and Main results

We consider the following quasilinear elliptic problem:

$$-\Delta u + \lambda u - \kappa \Delta(u^2)u = g(u) \text{ in } \mathbb{R}^N, \quad (1.1)$$

where  $\lambda > 0$ ,  $\kappa > 0$  and  $N \geq 2$ . Typical examples of the nonlinearity  $g(s)$  are given by  $g(s) = s^p$  for  $N \geq 3$  and  $g(s) = e^s - 1$  for  $N = 2$ . In this note, we review recent results on the uniqueness and the non-degeneracy of positive radial solutions of (1.1).

Equation (1.1) can be obtained as a stationary problem of the following modified Schrödinger equation:

$$i \frac{\partial z}{\partial t} = -\Delta z - \kappa \Delta(|z|^2)z - h(z), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \quad (1.2)$$

where  $z$  is a complex-valued function and  $h$  has the Gauge invariance, that is,  $h(e^{i\theta}z) = h(z)$  for all  $\theta \in \mathbb{R}^N$ . Equation (1.2) appears in the study of plasma physics. (See [6, 10] for the derivations.) Especially if we consider the standing wave of (1.2) of the form  $z(t, x) = u(x)e^{i\lambda t}$ , then  $u(x)$  satisfies (1.1) provided  $g(s) = h(s) - \lambda s$ .

Equation (1.1) has a variational structure, that is, one can obtain solutions of (1.1) as critical points of the associated functional  $I$  defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2\kappa u^2) |\nabla u|^2 + \lambda u^2 dx - \int_{\mathbb{R}^N} G(u) dx,$$

where  $G(s) = \int_0^s g(t) dt$ . In applications, the most important solution is the so-called *ground state*, which is a solution of (1.1) having the least energy among all non-trivial solutions. When we study the stability of the standing wave, the uniqueness and the non-degeneracy of the ground state play an important role.

As for the existence of ground states, we have the following result.

**Theorem 1.1** [1, 8]. *Let  $\lambda > 0$ ,  $\kappa > 0$  and suppose  $g(s) = s^p$ ,  $1 < p < \frac{3N+2}{N-2}$  for  $N \geq 3$  and  $g(s) = e^s - 1$  for  $N = 2$ . Then there exists a ground state of (1.1). Moreover any ground state is of the class  $C^2(\mathbb{R}^N)$ , positive, radially symmetric and decreasing with respect to  $r = |x|$  (up to translation).*

**Remark 1.2.** *We can obtain the existence of a ground state for more general nonlinearities. (See [4, 5] for details.)*

By Theorem 1.1, we can see that if we could show the uniqueness and the non-degeneracy of positive radial solutions of (1.1), then the ground state of (1.1) is also unique and non-degenerate. However, the uniqueness and the non-degeneracy of positive solutions of (1.1) seem to be difficult and are less studied. In [2, 4, 5], they proved the uniqueness and non-degeneracy if  $\kappa$  is sufficiently small by applying the perturbation method. In this note, we show the uniqueness and the non-degeneracy of the positive radial solution for another range of parameters  $\lambda$  and  $\kappa$ . Indeed, we have the following result.

**Theorem 1.3.**

- (i) *Suppose  $N \geq 3$ ,  $g(s) = s^p$  and  $1 < p < \frac{3N+2}{N-2}$ . There exists  $c_0 = c_0(p) > 0$  such that if  $\kappa\lambda^{\frac{2}{p-1}} \geq c_0$ , then (1.1) has a unique positive radial solution.*
- (ii) *Suppose  $N = 2$ ,  $\kappa > 0$  and  $g(s) = e^s - 1 - s$ . There exists  $\lambda^* > 0$  independent of  $\kappa$  such that if  $\lambda \geq \lambda^*$ , then (1.1) has a unique positive radial solution.*

**Theorem 1.4.** *Under the assumptions of Theorem 1.3, the kernel of the linearized operator around the unique positive radial solution  $w$  is given by*

$$\text{Ker}(L) = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_N} \right\}.$$

*Epecially  $w$  is non-degenerate in  $H_{rad}^1(\mathbb{R}^N)$ , that is, if  $L(\phi) = 0$  and  $\phi \in H_{rad}^1(\mathbb{R}^N)$ , then  $\phi \equiv 0$ . Here the linearized operator  $L$  of (1.1) defined by*

$$L(\phi) = -\Delta\phi + \lambda\phi - g'(w)\phi - 2\kappa \text{div}(w^2\nabla\phi) - \kappa(4w\Delta w + 2w|\nabla w|^2)\phi.$$

**Remark 1.5.** *Theorem 1.3 (i) means that if either  $\kappa$  or  $\lambda$  is sufficiently large, then the uniqueness holds. On the other hand in Theorem 1.3 (ii), the uniqueness holds only when  $\lambda$  is sufficiently large. In the case  $g(s) = s^p$ , we have a nice scaling. Namely for a solution  $u$  of (1.1), we rescale  $\tilde{u}(x)$  as  $u(x) = \lambda^{\frac{1}{p-1}}\tilde{u}(\lambda^{\frac{1}{2}}x)$ . Then we can see that (1.1) is reduced to*

$$-\Delta\tilde{u} + \tilde{u} - \kappa\lambda^{\frac{2}{p-1}}\Delta(\tilde{u}^2)\tilde{u} = \tilde{u}^p \quad \text{in } \mathbb{R}^N.$$

Thus in the case  $g(s) = s^p$ , we can describe the condition for the uniqueness in terms of  $\kappa\lambda^{\frac{2}{p-1}}$ . In the case  $g(s) = e^s - 1$ , such a scaling seems not to work well.

We prove Theorems 1.3-1.4 by applying the shooting method. However since equation (1.1) is quasilinear, it seems to be difficult to consider (1.1) directly. To avoid this difficulty, we adapt *dual approach* as in [1, 7, 12]. More precisely, we convert our quasilinear equation into a semilinear equation by using a suitable translation  $f$ . We will see that the set of positive radial solutions has one-to-one correspondence to that of the semilinear problem. This enables us to apply uniqueness results [15, 16, 17] for semilinear elliptic equations. We will also see in Lemma 2.3 and Proposition 2.4 below, there is a strong relation between the linearized operator of the original quasilinear equation and that of the converted semilinear equation. By this relation, we have only to study the non-degeneracy for the semilinear problem.

## 2. Dual approach

In this section, we introduce a dual variational formulation of (1.1). Firstly we study some properties of the unique solution of the ODE related to (1.1). As we will see later, this unique solution gives one-to-one correspondence between (1.1) and a semilinear elliptic problem (2.2) below.

Let  $f(t)$  be a solution of the following ODE:

$$f'(t) = \frac{1}{\sqrt{1 + 2\kappa f(t)^2}} \text{ on } [0, \infty), \quad f(0) = 0. \quad (2.1)$$

For  $t < 0$ , we put  $f(t) = -f(-t)$ . By the standard theory of ODE, we can see that  $f$  is uniquely determined, of class  $C^2$  and invertible on  $\mathbb{R}$ .

From (2.1), we can show the following.

**Lemma 2.1** [1].  *$f(t)$  satisfies the following properties:*

- (i)  $0 \leq f(t) \leq t$ ,  $0 < f'(t) \leq 1$  for all  $t \geq 0$ .  $t \leq f(t) \leq 0$ ,  $0 < f'(t) \leq 1$  for all  $t \leq 0$ .
- (ii)  $f''(t) = \frac{1}{f(t)}(f'(t)^4 - f'(t)^2)$  for  $t > 0$ .
- (iii)  $\frac{1}{2}f(t) \leq f'(t)t \leq f(t)$  for all  $t \geq 0$ .
- (iv)  $\lim_{s \rightarrow 0} \frac{f(s)}{s} = 1$ .

Using the function  $f(t)$ , we consider the following semilinear elliptic problem, which we call the *dual problem*:

$$-\Delta v + \lambda f(v)f'(v) = g(f(v))f'(v) \text{ in } \mathbb{R}^N. \quad (2.2)$$

Then we have the following relation between (1.1) and (2.2).

**Proposition 2.2** [1].  $u \in X \cap C^2(\mathbb{R}^N)$  is a positive radial solution of (1.1) if and only if  $v = f^{-1}(u) \in H^1 \cap C^2(\mathbb{R}^N)$  is a positive radial solution of (2.2).

Proposition 2.2 tells us that if (2.2) has a unique positive radial solution  $\tilde{w}$ , then  $w = f(\tilde{w})$  is a unique positive radial solution of (1.1). Thus we have only to study the uniqueness of the positive radial solution of the semilinear problem (2.2).

In order to study the non-degeneracy of the unique positive radial solution, we need more detailed correspondence between (1.1) and (2.2).

To state the result, let  $\tilde{L} : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$  be a linearized operator around  $\tilde{w}$  of (2.2), which is defined by

$$\begin{aligned} \tilde{L}(\psi) := & -\Delta\psi + \lambda \left( f'(\tilde{w})^2 + f(\tilde{w})f''(\tilde{w}) \right) \psi \\ & - \left( g'(f(\tilde{w})) f'(\tilde{w})^2 + g(f(\tilde{w}))f''(\tilde{w}) \right) \psi. \end{aligned} \quad (2.3)$$

Then we have the following.

**Lemma 2.3.** Suppose that  $w \in H^1 \cap C^2(\mathbb{R}^N)$  is a positive solution of (1.1) and put  $\tilde{w} = f^{-1}(w)$ . Let  $L$  and  $\tilde{L} : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$  be the linearized operators defined by (1.4) and (2.3) respectively. Finally for  $\phi \in H^2(\mathbb{R}^N)$ , we put  $\psi = \sqrt{1 + 2\kappa w^2}\phi$ . Then it follows that

$$\tilde{L}(\psi) = \frac{1}{\sqrt{1 + 2\kappa w^2}} L(\phi). \quad (2.4)$$

**Proof.** By direct computations, we have

$$\nabla\psi = \frac{2\kappa w\phi}{\sqrt{1 + 2\kappa w^2}} \nabla w + \sqrt{1 + 2\kappa w^2} \nabla\phi,$$

and

$$\begin{aligned} \Delta\psi &= \nabla \left( \frac{2\kappa w\phi}{\sqrt{1 + 2\kappa w^2}} \right) \cdot \nabla w + \frac{2\kappa w\phi}{\sqrt{1 + 2\kappa w^2}} \Delta w \\ &\quad + \nabla \left( \sqrt{1 + 2\kappa w^2} \right) \cdot \nabla\phi + \sqrt{1 + 2\kappa w^2} \Delta\phi \\ &= \sqrt{1 + 2\kappa w^2} \Delta\phi + \frac{4\kappa w}{\sqrt{1 + 2\kappa w^2}} \nabla w \cdot \nabla\phi + \frac{2\kappa |\nabla w|^2}{(\sqrt{1 + 2\kappa w^2})^3} \phi + \frac{2\kappa w \Delta w}{\sqrt{1 + 2\kappa w^2}} \phi. \end{aligned}$$

Next by Lemma 2.1 (ii) and from (2.1), we get

$$(f'(\tilde{w})^2 + f(\tilde{w})f''(\tilde{w})) \psi = f'(\tilde{w})^4 \psi = \frac{1}{(\sqrt{1 + 2\kappa w^2})^3} \phi,$$

and

$$\begin{aligned}
& \left( g'(f(\tilde{w}))f'(\tilde{w})^2 + g(f(\tilde{w}))f''(\tilde{w}) \right) \psi \\
&= g'(f(\tilde{w}))f'(\tilde{w})^2 \psi + g(f(\tilde{w})) \frac{f'(\tilde{w})^4 - f'(\tilde{w})^2}{f(\tilde{w})} \psi \\
&= \frac{g'(f(\tilde{w}))}{\sqrt{1+2\kappa w^2}} \phi - \frac{2\kappa w}{(\sqrt{1+2\kappa w^2})^3} g(f(\tilde{w})) \phi.
\end{aligned}$$

Thus from (1.1), (1.4) and (2.3), we obtain

$$\begin{aligned}
\tilde{L}(\psi) &= -\Delta \psi + \lambda(f'^2 + f f'') \psi - (g'(f(\tilde{w}))f'^2 + g(f(\tilde{w}))f'') \psi \\
&= -\sqrt{1+2\kappa w^2} \Delta \phi - \frac{4\kappa w}{\sqrt{1+2\kappa w^2}} \nabla w \cdot \nabla \phi - \frac{2\kappa |\nabla w|^2}{(\sqrt{1+2\kappa w^2})^3} \phi \\
&\quad - \frac{2\kappa w \Delta w}{\sqrt{1+2\kappa w^2}} \phi + \frac{\lambda}{(\sqrt{1+2\kappa w^2})^3} \phi - \frac{g'(w)}{\sqrt{1+2\kappa w^2}} \phi + \frac{2\kappa w}{(\sqrt{1+2\kappa w^2})^3} g(w) \phi \\
&= \frac{1}{\sqrt{1+2\kappa w^2}} \left( -(1+2\kappa w^2) \Delta \phi - 4\kappa w \nabla w \cdot \nabla \phi - \frac{2\kappa |\nabla w|^2}{1+2\kappa w^2} \phi \right. \\
&\quad \left. - 2\kappa w \Delta w \phi + \frac{\lambda}{1+2\kappa w^2} \phi - g'(w) \phi + \frac{2\kappa w}{1+2\kappa w^2} g(w) \phi \right) \\
&= \frac{1}{\sqrt{1+2\kappa w^2}} L(\phi) \\
&\quad + \frac{2\kappa w}{(\sqrt{1+2\kappa w^2})^3} (\Delta w - \lambda w + 2\kappa w |\nabla w|^2 + 2\kappa w^2 \Delta w + g(w)) \phi \\
&= \frac{1}{\sqrt{1+2\kappa w^2}} L(\phi).
\end{aligned}$$

This completes the proof. ■

By Lemma 2.3, we obtain the following result on the linearized operators.

**Proposition 2.4.** *Suppose that  $w \in H^1 \cap C^2(\mathbb{R}^N)$  is a positive solution of (1.1) and put  $\tilde{w} = f^{-1}(w)$ . Then*

- (i)  $\phi \in \text{Ker}(L)$  if and only if  $\psi = \sqrt{1+2\kappa w^2} \phi \in \text{Ker}(\tilde{L})$ .
- (ii)  $w$  is non-degenerate if and only if  $\tilde{w}$  is non-degenerate.
- (iii)  $\text{Ker}(L) = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_N} \right\}$  if and only if  $\text{Ker}(\tilde{L}) = \text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\}$

**Proof.** (i) From (2.4), it follows that

$$\tilde{L}(\psi) = 0 \Leftrightarrow L(\phi) = 0.$$

Thus the claim holds.

(ii) The claim follows from (i).

(iii) We assume that  $\text{Ker}(L) = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_N} \right\}$ . Suppose by contradiction that  $\text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\} \neq \text{Ker}(\tilde{L})$ . Since  $\frac{\partial \tilde{w}}{\partial x_i} \in \text{Ker}(\tilde{L})$  for  $i = 1, \dots, N$ , it follows that

$$\text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\} \subseteq \text{Ker}(\tilde{L}).$$

Thus there exists  $\psi \neq 0$  such that

$$\psi \in \text{Ker}(\tilde{L}) \setminus \text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\}.$$

Since  $\psi \in \text{Ker}(\tilde{L})$ , we have  $\tilde{L}(\psi) = 0$ . Putting  $\psi = \sqrt{1 + 2\kappa w^2} \phi$ , we obtain  $L(\phi) = 0$  by Lemma 2.3. Then by the assumption  $\text{Ker}(L) = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_N} \right\}$ , there exist  $c_1, \dots, c_N$  such that

$$\phi = c_1 \frac{\partial w}{\partial x_1} + \dots + c_N \frac{\partial w}{\partial x_N}.$$

Now since  $w = f(\tilde{w})$ , it follows that

$$\frac{\partial w}{\partial x_i} = f'(\tilde{w}) \frac{\partial \tilde{w}}{\partial x_i} = \frac{1}{\sqrt{1 + 2\kappa w^2}} \frac{\partial \tilde{w}}{\partial x_i} \text{ for } i = 1, \dots, N.$$

Thus we have

$$\psi = c_1 \frac{\partial \tilde{w}}{\partial x_1} + \dots + c_N \frac{\partial \tilde{w}}{\partial x_N} \in \text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\}.$$

This is a contradiction and hence  $\text{Ker}(\tilde{L}) = \text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\}$ .

We can show the converse in a similar way. ■

By Proposition 2.4, we have only to study the non-degeneracy of the unique positive radial solution of the semilinear problem (2.2).

### 3. Uniqueness of the positive radial solution

In this section, we study the uniqueness of the positive radial solutions (2.2). For simplicity, we put

$$h(s) = g(f(s))f'(s) - \lambda f(s)f'(s) \text{ for } s \geq 0. \quad (3.1)$$

We distinguish the cases  $N \geq 3$  and  $N = 2$ .

#### 3.1. Uniqueness for $N \geq 3$

In this case, we suppose that  $g(s) = s^p$ ,  $1 < p < \frac{3N+2}{N-2}$ . We apply the following uniqueness result due to [17].

**Proposition 3.1** [17]. Suppose that there exists  $b > 0$  such that

- (i)  $h$  is continuous on  $(0, \infty)$ ,  $h(s) \leq 0$  on  $(0, b]$  and  $h(s) > 0$  for  $s > b$ .
- (ii)  $g \in C^1(b, \infty)$  and  $\frac{d}{ds} \left( \frac{sh'(s)}{h(s)} \right) \leq 0$  on  $(b, \infty)$ .

Then the semilinear problem:

$$-\Delta u = h(u) \text{ in } \mathbb{R}^N, \quad u > 0, \quad u \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad u(0) = \max u(x)$$

has at most one positive radial solution.

Now we can see that  $h$  defined in (3.1) is of the class  $C^1[0, \infty)$  and

$$h(s) = 0 \iff f^{p-1}(s) = \lambda \iff s = f^{-1}(\lambda^{\frac{1}{p-1}}).$$

We put  $b := f^{-1}(\lambda^{\frac{1}{p-1}})$ . Since  $(s - b)g(s) = (s - b)f f'(f^{p-1} - \lambda)$ , we can see (i) of Proposition 3.1 holds. From (2.1), we can also observe that

$$f'(b) = \frac{1}{\sqrt{1 + 2\kappa\lambda^{\frac{2}{p-1}}}}.$$

Since  $f'(s) \rightarrow 0$  as  $s \rightarrow \infty$ , this implies

$$b \rightarrow \infty \text{ if and only if } \kappa\lambda^{\frac{2}{p-1}} \rightarrow \infty. \quad (3.2)$$

**Lemma 3.2** [1]. There exists  $c_0 = c_0(p) > 0$  such that if  $\kappa\lambda^{\frac{2}{p-1}} \geq c_0$ , then  $h$  satisfies (ii) of Proposition 3.1.

### 3.2. Uniqueness for $N = 2$

In this case, we suppose that  $g(s) = e^s - 1$ . We apply the following uniqueness result due to Pucci-Serrin [15, 16].

**Proposition 3.3** ([15, 16]). Suppose that the function  $h(s)$  satisfies the following assumptions:

- (i)  $h$  is continuous on  $[0, \infty)$  and  $h(0) = 0$ .
- (ii)  $h$  is continuously differentiable on  $(0, \infty)$ .
- (iii) There exists  $s_0 > 0$  such that  $h(s_0) = 0$  and

$$\begin{cases} h(s) < 0 & \text{for } 0 < s < s_0, \\ h(s) > 0 & \text{for } s_0 < s < \infty. \end{cases}$$



(iv)  $\frac{d}{ds} \left( \frac{H(s)}{h(s)} \right) \geq 0$  for  $s > 0$ ,  $s \neq s_0$ . Here  $H(s) = \int_0^s h(t) dt$ .

Then the semilinear problem:

$$-\Delta u = h(u) \text{ in } \mathbb{R}^2, \quad u > 0, \quad u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad u(0) = \max u(x)$$

has at most one positive radial solution.

Now we can see that the function  $h(s)$  defined in (3.1) satisfies (i) and (ii). Moreover since  $f'(s) \neq 0$  for all  $s > 0$ , there exists a unique  $s_0 > 0$  such that

$$h(s_0) = (e^{f(s_0)} - 1 - \lambda f(s_0))f'(s_0) = 0,$$

$$h(s) < 0 \text{ for } 0 < s < s_0 \text{ and } h(s) > 0 \text{ for } s_0 < s < \infty.$$

Thus it remains to show that (iv) holds.

**Lemma 3.4 [3].** *There exists  $\lambda^* > 0$  independent of  $\kappa > 0$  such that for any  $\lambda > \lambda^*$  and  $\kappa > 0$ , it follow that*

$$\frac{d}{ds} \left( \frac{H(s)}{h(s)} \right) \geq 0 \text{ for all } s > 0, s \neq s_0.$$

By Theorem 1.1, Propositions 3.1, 3.3 and Lemmas 3.2, 3.4, we obtain the uniqueness result.

**Proposition 3.5.**

- (i) Suppose  $N \geq 3$ ,  $g(s) = s^p$  and  $1 < p < \frac{3N+2}{N-2}$ . There exists  $c_0 = c_0(p) > 0$  such that if  $\kappa \lambda^{\frac{2}{p-1}} \geq c_0$ , then (2.2) has a unique positive radial solution.
- (ii) Suppose  $N = 2$ ,  $\kappa > 0$  and  $g(s) = e^s - 1 - s$ . There exists  $\lambda^* > 0$  independent of  $\kappa$  such that if  $\lambda \geq \lambda^*$ , then (2.2) has a unique positive radial solution.

#### 4. Non-degeneracy of the unique positive radial solution

In this section, we show that the unique positive radial solution of (2.2) is non-degenerate. We argue as in [9]. To this aim, we study the structure of radial solutions of the following ODE:

$$\begin{cases} v'' + \frac{N-1}{r}v' + \hat{g}(v) = 0, & r \in (0, \infty), \\ v(0) = d > 0. \end{cases} \quad (4.1)$$

Here we denote  $' = \frac{d}{dr}$  and

$$\hat{g}(s) = g(f(s))_+ f'(s) - (\lambda - 1)f(s)f'(s). \quad (4.2)$$

Then we can see that for each  $d > 0$ , (4.1) has a solution  $v(r, d)$ .

As in [11], we classify each  $d > 0$  as follows:

$$N = \{d > 0; \text{there exists } r_0 = r_0(d) \in (0, \infty) \text{ such that } v(r_0, d) = 0\}.$$

$$G = \{d > 0; v(r, d) > 0 \text{ for all } r > 0 \text{ and } \lim_{r \rightarrow \infty} v(r, d) = 0\}.$$

$$P = \{d > 0; v(r, d) > 0 \text{ for all } r > 0 \text{ but } \liminf_{r \rightarrow \infty} v(r, d) > 0\}.$$

First we prove the following properties on  $N$ .

**Lemma 4.1.**  *$N$  satisfies the following properties:*

- (i) *There exists  $\hat{d} > 0$  such that  $v(r, \hat{d})$  has a finite zero. Especially it follows that  $N \neq \emptyset$ .*
- (ii)  *$N$  is an open set.*
- (iii) *For  $d \in N$ , it follows that  $v(r, d) \rightarrow -\infty$  as  $r \rightarrow \infty$ .*

**Proof.** (i) Let  $R > 0$  be arbitrarily given. We consider the auxiliary problem:

$$\begin{cases} -\Delta v = \hat{g}(v) & \text{in } B_R(0). \\ v > 0 & \text{in } B_R(0). \\ v = 0 & \text{on } \partial B_R(0). \end{cases} \quad (4.3)$$

Then we can show that (4.3) has a positive radial solution  $v_R(x)$ . Putting  $\hat{d} = v_R(0)$ , we obtain  $v(R, \hat{d}) = 0$  for a solution of (4.1).

(ii) The claim follows from the continuous dependence on the initial value. (see [11] Lemma 13, P. 253.)

(iii) For  $d > 0$ , let  $r_0 = r_0(d) > 0$  be the first zero of  $v(r) = v(r, d)$ . Then we have  $v'(r_0) < 0$ .

Suppose that there exists  $r_1 > r_0$  such that  $v(r_1) < 0$  and  $v'(r_1) = 0$ . Then from Lemma 2.1 (i), (4.1) and (4.2), we have

$$v''(r_1) = -\hat{g}(v(r_1)) = (\lambda - 1)f(v(r_1))f'(v(r_1)) < 0.$$

Thus  $v(r)$  can not take a negative local minimum for  $r > r_0$ . This implies that  $v(r)$  does not converge to zero as  $r \rightarrow \infty$  and  $v(r)$  does not oscillate at infinity.

Next we suppose by contradiction that there exists  $c < 0$  such that  $v(r) \rightarrow c < 0$  as  $r \rightarrow \infty$ . Then we have  $v'(r) \rightarrow 0$  as  $r \rightarrow \infty$ . Since  $\hat{g}(s) > 0$  for  $s < 0$ , it follows from (4.1) that  $v''(r) < M < 0$  for sufficiently large  $r$  and some  $M < 0$ . This contradicts to the fact  $v(r) \rightarrow c < 0$  as  $r \rightarrow \infty$ . Thus we obtain  $v(r) \rightarrow -\infty$  as  $r \rightarrow \infty$ . ■

Next we show the following result on  $P$ .

**Lemma 4.2.** *P satisfies the following properties:*

- (i) *Let  $s_1 > 0$  be a unique zero of  $\hat{G}(s)$ , where  $\hat{G}(s) = \int_0^s \hat{g}(t) dt$ . Then for any  $d \leq s_1$ , it follows that  $d \in P$ . Especially we have  $(0, s_1] \subset P$ .*
- (ii) *P is an open set.*

**Proof.** (i) We define the energy  $E$  by

$$E(r) = E(v(r, d)) := \frac{1}{2}(v'(r))^2 + \hat{G}(v(r)), \quad (4.4)$$

Then from (4.1), we have

$$E'(r) = -\frac{N-1}{r}(v'(r))^2 < 0.$$

Now we take  $d \leq s_1$ . Then it follows from  $v(0) = d$  and  $v'(0) = 0$  that  $E(0) = \hat{G}(d)$ . Since  $\hat{G}(s) \leq 0$  for  $0 \leq s \leq s_1$ , we get

$$E(r) < E(0) \leq 0 \text{ for all } r > 0. \quad (4.5)$$

Next we prove that  $s_1 \notin N \cup G$ . First we show that  $v(r, s_1)$  does not have a finite zero. To this aim, suppose by contradiction that  $v(r_0) = 0$  for some  $r_0 > 0$ . Then from  $\hat{G}(0) = 0$  and (4.4), it follows that  $E(r_0) = \frac{1}{2}(v'(r_0))^2 > 0$ . This contradicts to (4.5).

Finally we show that  $v(r, s_1)$  does not converges to zero as  $r \rightarrow \infty$ . If  $v(r) \rightarrow 0$  as  $r \rightarrow \infty$ , then  $v(r)$  decays exponentially up to the first derivative. Thus it follows that  $E(r) \rightarrow 0$  as  $r \rightarrow \infty$ . This is a contradiction.

(ii) By the continuous dependence of the initial value, the conclusion holds. ■

Now by Proposition 3.5, we know that the positive radial solution of (2.2) is unique. This implies that there exists  $d^* > 0$  such that  $G = \{d^*\}$ . Moreover by the proof of Lemma 4.2, we can see that  $s_1 < d^*$ . Since  $N$  and  $P$  are open, we obtain the following structure.

**Proposition 4.3.** *There exists a unique  $d^* > 0$  such that*

$$N = (d^*, \infty), \quad G = \{d^*\} \text{ and } P = (0, d^*).$$

In order to prove the non-degeneracy, we define the Pohozaev value  $P$  by

$$P(r) = P(r; v(r, d)) := \frac{r^N}{2}(v'(r))^2 + r^N \hat{G}(v(r)).$$

Then from (4.1), we obtain the Pohozaev type identity:

$$\frac{d}{dr} P(r) = -\frac{N-2}{2} r^{N-1} (v'(r))^2 + N r^{N-1} \hat{G}(v(r)). \quad (4.6)$$

Moreover we have the following.

**Lemma 4.4.** *It follows that*

$$\lim_{r \rightarrow \infty} P(r; v(r, d)) = \begin{cases} 0 & \text{for } d = d^* \\ +\infty & \text{for } d > d^*. \end{cases}$$

**Proof.** If  $d = d^*$ , then  $v(r, d^*)$  and  $v'(r, d^*)$  decay exponentially as  $r \rightarrow \infty$ . Thus we can see that the claim holds.

For  $d > d^*$ , we have  $v(r, d) \rightarrow -\infty$  as  $r \rightarrow \infty$  by Lemma 4.1 (iii) and Proposition 4.3. From (4.2), it follows that  $\hat{G}(s) = \frac{\lambda-1}{2} f(s)^2$  for  $s < 0$  and hence  $\hat{G}(s) \rightarrow +\infty$  as  $s \rightarrow -\infty$ . Thus we have  $P(r; v(r, d)) \rightarrow +\infty$  for  $d > d^*$ . ■

Next we consider the linearized equation of (4.1):

$$\begin{cases} \phi'' + \frac{N-1}{r} \phi' + \hat{g}'(v) \phi = 0, & r \in (0, \infty). \\ \phi(0) = 1, \phi'(0) = 0. \end{cases} \quad (4.7)$$

Since  $\frac{\partial v}{\partial d}(r, d^*)$  satisfies (4.7),  $\frac{\partial v}{\partial d}$  can be written by a constant multiple of  $\phi$ . Moreover we have the following.

**Proposition 4.5.**  $\frac{\partial v}{\partial d}(r, d^*)$  does not belong to  $H^1(\mathbb{R}^N)$ .

**Proof.** Suppose by contradiction that  $\frac{\partial v}{\partial d}(r, d^*) \in H^1(\mathbb{R}^N)$ .

Now from (4.6), we have

$$P(r; v(r, d)) = -\frac{N-2}{2} \int_0^r s^{N-1} (v'(s, d))^2 ds + N \int_0^r s^{N-1} \hat{G}(v(s, d)) ds.$$

Differentiating it with respect to  $d$ , we get

$$\begin{aligned} \frac{\partial}{\partial d} P(r; v(r, d)) &= -(N-2) \int_0^r s^{N-1} v' \left( \frac{\partial v}{\partial d} \right)' ds + N \int_0^r s^{N-1} \hat{g}(v) \frac{\partial v}{\partial d} ds \\ &= \left[ -(N-2) s^{N-1} v'(s, d) \frac{\partial v}{\partial d}(s, d) \right]_0^r \\ &\quad + (N-2) \int_0^r \left( (N-1) s^{N-2} v' + s^{N-1} v'' \right) \frac{\partial v}{\partial d} ds \\ &\quad + N \int_0^r s^{N-1} \hat{g}(v) \frac{\partial v}{\partial d} ds. \end{aligned}$$

From (4.1) and  $v'(0) = 0$ , it follows that

$$\frac{\partial}{\partial d} P(r; v(r, d)) = -(N-2) r^{N-1} v'(r, d) \frac{\partial v}{\partial d}(r, d) + 2 \int_0^r s^{N-1} \hat{g}(v) \frac{\partial v}{\partial d} ds.$$

Especially taking  $d = d^*$ , we obtain

$$\begin{aligned} \frac{\partial}{\partial d} P(r; v(r, d)) \Big|_{d=d^*} &= -(N-2)r^{N-1}v'(r, d^*) \frac{\partial v}{\partial d}(r, d^*) \\ &\quad + 2 \int_0^r s^{N-1} \hat{g}(v) \frac{\partial v}{\partial d}(s, d^*) ds. \end{aligned} \quad (4.8)$$

Moreover from (4.1) and (4.7), we also have

$$\begin{aligned} \left( r^N v' \left( \frac{\partial v}{\partial d} \right)' + r^N \hat{g}(v) \frac{\partial v}{\partial d} \right)' &= r^N \left( \frac{\partial v}{\partial d} \right)' \left( v'' + \frac{N-1}{r} v' + \hat{g}(v) \right) \\ &\quad + r^N v' \left( \left( \frac{\partial v}{\partial d} \right)'' + \frac{N-1}{r} \left( \frac{\partial v}{\partial d} \right)' + \hat{g}'(v) \frac{\partial v}{\partial d} \right) \\ &\quad - (N-2)r^{N-1}v' \left( \frac{\partial v}{\partial d} \right)' + Nr^{N-1} \hat{g}(v) \frac{\partial v}{\partial d} \\ &= -(N-2)r^{N-1}v' \left( \frac{\partial v}{\partial d} \right)' + Nr^{N-1} \hat{g}(v) \frac{\partial v}{\partial d}. \end{aligned}$$

Thus we obtain

$$\frac{\partial}{\partial d} P(r; v(r, d)) \Big|_{d=d^*} = r^N v' \left( \frac{\partial v}{\partial d} \right)' + r^N \hat{g}(v) \frac{\partial v}{\partial d}. \quad (4.9)$$

Next by the assumption, it follows that  $r^{\frac{N-1}{2}} \frac{\partial v}{\partial d}, r^{\frac{N-1}{2}} \left( \frac{\partial v}{\partial d} \right)' \in L^2(0, \infty)$ . Since  $v(r, d^*)$  and  $v'(r, d^*)$  decay exponentially as  $r \rightarrow \infty$ , we have from (4.9) that

$$\lim_{r \rightarrow \infty} \frac{\partial}{\partial d} P(r; v(r, d)) \Big|_{d=d^*} = 0. \quad (4.10)$$

Next let  $\phi$  be a solution of (4.7). We claim that  $\phi$  has a definite sign near infinity. First we observe that  $\hat{g}'(0) = -(\lambda - 1) < 0$ . Since  $v(r, d^*)$  decays exponentially as  $r \rightarrow \infty$ , there exists  $r_1 > 0$  such that  $\hat{g}'(v(r, d^*)) < 0$  for  $r > r_1$ .

Next we suppose that there exists  $r_2 > r_1$  such that  $\phi(r_1) > 0$  and  $\phi'(r_1) = 0$ . Then from (4.7), we have

$$\phi''(r_1) = -\frac{N-1}{r_1} \phi'(r_1) - \hat{g}'(v) \phi(r_1) > 0.$$

This means that  $\phi$  can not take a positive local maximum for  $r > r_1$ . Similarly we can see that  $\phi$  can not take a negative local minimum. Thus  $\phi$  has a constant sign for  $r > r_1$ . Hence it follows that either  $\frac{\partial v}{\partial d}(r, d^*) > 0$  or  $\frac{\partial v}{\partial d}(r, d^*) < 0$  for  $r > r_1$ .

If  $\frac{\partial v}{\partial d}(r, d^*) > 0$  for  $r > r_1$ , then  $v(r, d)$  is increasing with respect to  $d$  near  $d^*$ . Since  $v(r, d^*) > 0$ , it follows that  $v(r, d) > 0$  for  $d > d^*$  and  $r > r_1$ . By Lemma 4.1 (iii) and Proposition 4.3, this is a contradiction.

Finally suppose that  $\frac{\partial v}{\partial d}(r, d^*) < 0$  for  $r > r_1$ . Now from (4.8) and (4.10) and by the exponential decay of  $v'$ , we have

$$0 = \lim_{r \rightarrow \infty} \frac{\partial}{\partial d} P(r; v) \Big|_{d=d^*} = 2 \int_0^\infty s^{N-1} \hat{g}(v) \frac{\partial v}{\partial d} ds.$$

On the other hand since  $\hat{g}(v) < 0$  and  $\frac{\partial v}{\partial d} < 0$  for  $r > r_1$ , we also have

$$2 \int_r^\infty s^{N-1} \hat{g}(v) \frac{\partial v}{\partial d} ds > 0.$$

Thus from  $v' < 0$  and  $\frac{\partial v}{\partial d} < 0$ , it follows that

$$\frac{\partial}{\partial d} P(r; v) \Big|_{d=d^*} = -(N-2)r^{N-1}v' \frac{\partial v}{\partial d} + 2 \int_0^r s^{N-1} \hat{g}(v) \frac{\partial v}{\partial d} ds < 0 \text{ for } r > r_1.$$

This implies that  $P(r; v(r, d))$  is decreasing with respect to  $d$  near  $d^*$ . Thus for  $r > r_1$  and  $d > d^*$ , we obtain

$$P(r; v(r, d^*)) > P(r; v(r, d)).$$

However by Lemma 4.4, we know that  $P(r; v(r, d^*)) \rightarrow 0$  and  $P(r; v(r, d)) \rightarrow +\infty$  for  $d > d^*$  as  $r \rightarrow \infty$ . This is a contradiction.  $\blacksquare$

Proposition 4.5 implies that the unique positive radial solution  $\tilde{w}$  of (2.2) is non-degenerate in  $H_{rad}^1(\mathbb{R}^N)$ . Finally we show the following result on the linearized operator  $\tilde{L} = -\Delta + g'(\tilde{w})$  of (2.2).

**Proposition 4.6.** *The kernel of  $\tilde{L}$  is given by*

$$\text{Ker}(\tilde{L}) = \text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\}.$$

**Proof.** First we observe that  $\text{span}\{\frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N}\} \subset \text{Ker}(\tilde{L})$ . In fact, since  $\tilde{w}$  is a solution of (2.2),  $\frac{\partial \tilde{w}}{\partial x_i}$  satisfies

$$-\Delta \left( \frac{\partial \tilde{w}}{\partial x_i} \right) + g'(\tilde{w}) \frac{\partial \tilde{w}}{\partial x_i} = 0 \text{ in } \mathbb{R}^N, \quad i = 1, \dots, N.$$

Moreover by the elliptic regularity theory, we can see that  $\frac{\partial \tilde{w}}{\partial x_i} \in H^2(\mathbb{R}^N)$ . Thus it follows that  $\text{span}\{\frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N}\} \subset \text{Ker}(\tilde{L})$ .

To complete the proof, it suffices to show that  $\dim \text{Ker}(\tilde{L}) \leq N$ . To this aim, we apply the argument in [13, 18]. Suppose that  $\phi \in \text{Ker}(\tilde{L})$ , that is,  $\phi \in H^2(\mathbb{R}^N)$  and it satisfies

$$-\Delta \phi + g'(\tilde{w})\phi = 0 \text{ in } \mathbb{R}^N.$$

Then by the elliptic regularity theory, it follows that  $\phi \in C^2(\mathbb{R}^N)$ .

Now let  $\mu_i$  and  $\psi_i(\theta)$  with  $\theta \in S^{N-1}$  be the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on  $S^{N-1}$ , that is,

$$-\Delta_\theta \psi_i = \mu_i \psi_i.$$

Then it follows that

$$0 = \mu_0 < \mu_1 = \cdots = \mu_N = (N-1) < \mu_{N+1} \cdots$$

and  $\{\psi_i\}$  forms an orthonormal basis of  $L^2(S^{N-1})$ .

For  $\phi \in \text{Ker}(\tilde{L})$ , we define

$$\phi_i(r) := \int_{S^{N-1}} \phi(r, \theta) \psi_i(\theta) d\theta.$$

Then we have

$$\phi_i'' + \frac{N-1}{r} \phi_i' + \left( g'(\tilde{w}) - \frac{\mu_i}{r^2} \right) \phi_i = 0, \quad \phi_i'(0) = 0. \quad (4.11)$$

Moreover  $\phi \in \text{Ker}(\tilde{L})$  can be written as follows.

$$\phi(x) = \phi(r, \theta) = \sum_{i=0}^{\infty} \phi_i(r) \psi_i(\theta). \quad (4.12)$$

When  $i = 0$ , we have from  $\mu_0 = 0$  that

$$\phi_0'' + \frac{N-1}{r} \phi_0' + g'(\tilde{w}) \phi_0 = 0.$$

Then by Proposition 4.5, it follows that  $\phi_0 \equiv 0$ .

Next we show that  $\phi_i \equiv 0$  for  $i \geq N+1$ . If  $\phi_i \not\equiv 0$ , then  $\phi_i(0) \neq 0$  by the uniqueness of the ODE (4.11). Thus we may assume that  $\phi_i(0) > 0$ . Let  $r_i \in (0, \infty]$  be such that  $\phi_i(r) > 0$  on  $[0, r_i)$  and  $\phi_i(r_i) = 0$ .

First we suppose that  $r_i < \infty$ . Multiplying (4.11) by  $r^{N-1} \tilde{w}'$  and integrating it over  $[0, r_i]$ , we get

$$\int_0^{r_i} r^{N-1} \tilde{w}' \phi_i'' + (N-1) r^{N-2} \tilde{w}' \phi_i' + r^{N-1} g'(\tilde{w}) \tilde{w}' \phi_i - \mu_i r^{N-3} \tilde{w}' \phi_i dr = 0.$$

By the integration by parts, it follows that

$$r_i^{N-1} \tilde{w}'(r_i) \phi_i'(r_i) - \int_0^{r_i} r^{N-1} \tilde{w}'' \phi_i' dr + \int_0^{r_i} r^{N-1} g'(\tilde{w}) \tilde{w}' \phi_i - \mu_i r^{N-3} \tilde{w}' \phi_i dr = 0.$$

By the integration by parts again and combined with  $\phi(r_i) = 0$ , we obtain

$$r_i^{N-1} \tilde{w}'(r_i) \phi_i'(r_i) + \int_0^{r_i} (r^{N-1} \tilde{w}''' + (N-1)r^{N-2} \tilde{w}'' + r^{N-1} g'(\tilde{w}) \tilde{w}') \phi_i dr - \int_0^{r_i} \mu_i r^{N-3} \tilde{w}' \phi_i dr = 0.$$

Moreover since  $\tilde{w}$  satisfies (4.1), we have

$$\tilde{w}''' + \frac{N-1}{r} \tilde{w}'' - \frac{N-1}{r^2} \tilde{w}' + g'(\tilde{w}) \tilde{w}' = 0.$$

Thus we obtain

$$r_i^{N-1} \tilde{w}'(r_i) \phi_i'(r_i) + (N-1-\mu_i) \int_0^{r_i} r^{N-3} \tilde{w}' \phi_i dr = 0.$$

Since  $\tilde{w}'(r_i) < 0$  and  $\phi_i'(r_i) < 0$ , it follows that

$$(N-1-\mu_i) \int_0^{r_i} r^{N-3} \tilde{w}' \phi_i dr < 0.$$

On the other hand since  $\phi_i(r) > 0$  on  $(0, r_i)$  and  $\mu_i > N-1$  for  $i \geq N+1$ , we also have

$$0 < (N-1-\mu_i) \int_0^{r_i} r^{N-3} \tilde{w}' \phi_i dr.$$

This is a contradiction.

Next suppose that  $r_i = +\infty$ . Since  $\tilde{w}'(r)$  and  $\tilde{w}''(r)$  decay exponentially as  $r \rightarrow \infty$ , we have

$$(N-1-\mu_i) \int_0^\infty r^{N-3} \tilde{w}' \phi_i dr = 0.$$

This implies again that  $\phi_i \equiv 0$  for  $i \geq N+1$ .

Now since  $\phi_0 \equiv 0$  and  $\phi_i \equiv 0$  for  $i \geq N+1$ , we have from (4.12) that

$$\phi(x) = \phi(r, \theta) = \sum_{i=1}^N c_i \phi_i(r) \phi_i(\theta).$$

This means that  $\dim \text{Ker}(\tilde{L}) \leq N$  and hence  $\text{Ker}(\tilde{L}) = \text{span} \left\{ \frac{\partial \tilde{w}}{\partial x_1}, \dots, \frac{\partial \tilde{w}}{\partial x_N} \right\}$ . ■

## 5. Concluding remarks and open questions

In this note, we review recent results on the uniqueness and the non-degeneracy of positive radial solutions of (1.1).



When  $N \geq 3$ , the exponent  $\frac{3N+2}{N-2}$  appears naturally by applying the embedding  $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$  to  $u^2$ . Moreover we can see that  $p = \frac{3N+2}{N-2}$  is actually the critical exponent for the existence of nontrivial solutions. (See [1] for the detail.) As we have shown in Theorems 1.3-1.4, the uniqueness holds for  $1 < p < \frac{3N+2}{N-2}$ . This implies that  $p$  can be  $H^1$ -supercritical.

On the other hand when  $N = 2$ , we have shown the uniqueness only for the case  $g(s) = e^s - 1$ . By applying the Trudinger-Moser inequality to  $u^2$ , we can see that  $g(s)$  may have a faster growth like  $g(s) \sim e^{c_0 s^4}$  for some  $c_0 > 0$ . (See [14] for the detail.) Thus it is natural to ask "Can we show the uniqueness for the case  $g(s) \sim e^{c_0 s^4}$ ?" Unfortunately, we have no result even if  $g(s) = e^{s^2}$ .

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